Runge-Kutta Methods of order 2

(Midpoint method)

Recall:

Taylor methods of order 2:

$$\begin{cases} w_0 = \alpha \\ w_{i+1} = w_i + h\, T^{(2)}(t_i, w_i) \end{cases}$$

$$f(t_i, w_i) + \frac{h}{2} f'(t_i, w_i)$$

(error $\approx O(h^3)$)

Idea: Find a_1, α_1, β_1 s.t.

$$T^{(2)}(t_i, w_i) \approx a_1 f(t + \alpha_1, y + \beta_1)$$

and error stays $O(h^3)$

The derivation requires 2 Ingredients

- **Ingredient I**: Taylor polynomials for functions of 2 variables:

$$F(t+\alpha_1, y+\beta_1)$$

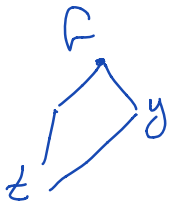
$$= F(t,y) + [\alpha_1 \;\; \beta_1] \begin{bmatrix} \frac{\partial f}{\partial t} \\ \frac{\partial f}{\partial y} \end{bmatrix}\Bigg|_{(t,y)} \quad (1)$$

$$+ \frac{1}{2}[\alpha_1 \;\; \beta_1] \begin{bmatrix} \frac{\partial^2 f}{\partial t^2} & \frac{\partial^2 f}{\partial t \partial y} \\ \frac{\partial^2 f}{\partial t \partial y} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix}\Bigg|_{(\xi,\mu)} \begin{bmatrix} \alpha_1 \\ \beta_1 \end{bmatrix}$$

error term "R_1" or "E_1"

where (ξ, μ) between (t,y) & $(t+\alpha_1, y+\beta_1)$

- **Ingredient II: Chain Rule**



$$f'(t,y) = \frac{df}{dt}(t, y(t))$$

$$= \frac{\partial f}{\partial t}(t,y) + \frac{\partial f}{\partial y}(t,y) \cdot \underbrace{\frac{dy}{dt}}_{y' = f}$$

and $T^{(2)}(t,y) = f(t,y) + \frac{h}{2} f'(t,y)$

so $T^{(2)}(t,y) = f + \frac{h}{2}\left[\frac{\partial f}{\partial t} + \frac{\partial f}{\partial y} \cdot f\right]$ (2)

Rewriting (1) & multiplying it by a_1:

$$a_1 f(t+\alpha_1, y+\beta_1) = a_1 f(t,y) + a_1\alpha_1 \frac{\partial f}{\partial t}(t,y) + a_1\beta_1 \frac{\partial f}{\partial y}(t,y) + a_1 R_1$$ (3)

Since we want $T^{(2)}(t,y) \approx a_1 f(t+\alpha_1, y+\beta_1)$

we just match coefficients of f, $\frac{\partial f}{\partial t}$ & $\frac{\partial f}{\partial y}$ in ② & ③ And we solve for a_1, α_1, β_1:

$$\Rightarrow \boxed{a_1 = 1}, \quad \boxed{\frac{h}{2} = a_1 \alpha_1}, \quad \boxed{\frac{h}{2} \cdot f(t,y) = a_1 \beta_1}$$

So now:

$$f\left(t + \frac{h}{2}, y + \frac{h}{2} f(t,y)\right) = T^{(2)}(t,y) + \underbrace{R_1\left(t + \frac{h}{2}, y + \frac{h}{2} f(t,y)\right)}$$

$$\left[\frac{h}{2} \quad \frac{h}{2} f(t,y)\right] \begin{bmatrix} \frac{\partial^2 f}{\partial t^2} & \frac{\partial^2 f}{\partial t \partial y} \\ \frac{\partial^2 f}{\partial t \partial y} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix} \begin{bmatrix} \frac{h}{2} \\ \frac{h}{2} f(t,y) \end{bmatrix}$$

$\approx O(h^2)$ if all partials are bounded.

$\Rightarrow$ local error is $O(h^3)$

So we replace $T^{(2)}(t,y)$ by

$f(t+h/2, y+h/2 f(t,y))$

to get:

$$\begin{cases} w_0 = \alpha \\ w_{i+1} = w_i + hf\left(t_i + \frac{h}{2}, w_i + \frac{h}{2} f(t_i, w_i)\right) \\ \text{for } i = 0, \dots, N-1 \end{cases}$$

↳ Runge-Kutta method of order 2

↳ also known as: Midpoint method

Another $O(h^3)$ method is given by

$$\begin{cases} w_0 = \alpha \\ w_{i+1} = w_i + \frac{h}{2}\left[f(t_i, w_i) + f(t_{i+1}, w_i + hf(t_i, w_i))\right] \end{cases}$$

$f(t_i, w_i)$: y' at (t_i, w_i)

$f(t_{i+1}, w_i + hf(t_i, w_i))$: est. of y' at $(t_{i+1}, w_i + hf(t_i, w_i))$

$w_i + hf(t_i, w_i)$: 1st order approx of w_{i+1}

modified Euler Method / Heun's Method

≈ Average of derivatives/slopes

Idea behind this modification:

$w_i + h f(t_i, w_i) =: \tilde{w}_{i+1}$ is the estimate from Euler's method of y_{i+1}.

Rather than use $\tilde{w}_{i+1}$ directly, we plug it back into

$f(t_{i+1}, \tilde{w}_{i+1})$ ≈ estimate of slope at t_{i+1}

and average it with $f(t_i, w_i)$, to get

$$\begin{cases} w_{i+1} = w_i + \frac{h}{2}\left[f(t_i, w_i) + f(t_{i+1}, \tilde{w}_{i+1})\right] \\ \text{where } \tilde{w}_{i+1} = w_i + h f(t_i, w_i) \end{cases}$$

General RK2:

$$w_{i+1} = w_i + a_1 h f(t_i, w_i) + a_2 h f(t_i + \alpha, y + \beta)$$

Example of Higher order RK method:

RK order 4:

$$
\begin{cases}
w_0 = \alpha \\
k_1 = h f(t_i, w_i) \\
k_2 = h f\left(t_i + h/2,\ w_i + \frac{1}{2}k_1\right) \\
k_3 = h f\left(t_i + h/2,\ w_i + \frac{1}{2}k_2\right) \\
k_4 = h f\left(t_{i+1},\ w_i + k_3\right) \\
w_{i+1} = w_i + \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)
\end{cases}
$$

est of y' at $t_i + h/2$

Error $= O(h^5)$

(provided $y \in C^{(5)}$)

Summary of our numerical methods for IVP's:

$$\begin{cases} y' = f(t,y), & t \in [a,b] \\ y(a) = \alpha \end{cases}$$

(I) Euler's method $O(h)$

$$w_0 = \alpha$$
$$w_{i+1} = w_i + hf(t_i, w_i)$$

(II) Higher order Taylor methods $O(h^{n+1})$

$$w_0 = \alpha$$
$$w_{i+1} = w_i + h\,T^{(n)}(t_i, w_i)$$

where $T^{(n)}(t_i,w_i) = f(t_i,w_i) + \frac{h}{2}f'(t_i,w_i) + \frac{h^2}{6}f''(t_i,w_i) + \cdots + \frac{h^{n-1}}{n!}f^{(n-1)}(t_i,w_i)$

(III) Runge-Kutta methods

(IIIa) Order 2 ($O(h^3)$)

$$w_0 = \alpha$$
$$w_{i+1} = w_i + hf\left(t_i + \frac{h}{2}, w_i + \frac{h}{2} f(t_i, w_i)\right)$$

(IIIb) Order 4 ($O(h^5)$)

(See before)

(IV) Other modifications

e.g. modified Euler

$$w_0 = \alpha$$
$$w_{i+1} = w_i + \frac{h}{2}\left(f(t_i, w_i) + f(t_{i+1}, \tilde{w}_{i+1})\right)$$

where

$$\tilde{w}_{i+1} = w_i + hf(t_i, w_i)$$